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Towards deterministic equations for Lévy walks: The fractional material derivative

Igor M. Sokolov[1,*] and Ralf Metzler[2,†]
[1]*Institut für Physik, Humboldt-Universität zu Berlin, Invalidenstraße 110, 10115 Berlin, Germany*
[2]*NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

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Lévy walks are random processes with an underlying spatiotemporal coupling. This coupling penalizes long jumps, and therefore Lévy walks give a proper stochastic description for a particle's motion with broad jump length distribution. We derive a generalized dynamical formulation for Lévy walks, in which the *fractional* equivalent of the *material derivative* occurs. Our approach is expected to be useful for the dynamical formulation of Lévy walks in an external force field or in phase space, for which the description in terms of the continuous time random walk or its corresponding generalized master equation are less well suited.

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Anomalous diffusion processes are characterized by deviations from the traditional linear time dependence $\langle x^2(t)\rangle = 2Kt$ of the mean squared displacement in the force-free limit. In particular, one distinguishes subdiffusion ($0<\kappa<1$) and superdiffusion ($\kappa>1$) for the wide class of systems displaying a power-law anomaly $\langle x^2(t)\rangle = 2K_\kappa t^\kappa/\Gamma(1+\kappa)$; here, K_κ is a generalized diffusion constant [1,2]. A versatile framework for the description of anomalous diffusion are continuous time random walks (CTRWs), which define a random walk that is governed by two probability density functions (pdfs), the jump length and waiting time distributions $\lambda(x)$ and $\psi(t)$ from which the jump length x and the waiting time t of each jump are drawn [3]. Although the stochastic formulation of the CTRW fully defines the random process and leads to the closed integral equation for the pdf of the particle's position $P(x,t)$ in terms of $\lambda(x)$ and $\psi(t)$, its mathematical handling gets awkward as soon as non-natural boundary conditions, the presence of external force fields, or the description in phase space are considered. The same complication holds true for the formulation in terms of generalized master equations, which are equivalent to CTRWs with uncorrelated $\lambda(x)$ and $\psi(t)$ [4]. In such cases, the corresponding deterministic equations of the generalized Fokker-Planck type, in which the drift terms occur explicitly and which can be attacked with the standard mathematical tools, render a much more amenable description. To find such equations for anomalous transport statistics has been a focal point in stochastic systems studies [5].

For subdiffusion processes, a complete framework of transport equations has been established, namely, the fractional Fokker-Planck and Klein-Kramers equations [5–7]. These are natural generalizations of their Brownian counterparts, and their solution exists, whenever the solution of the corresponding regular Fokker-Planck equation exists, as they correspond to a *subordination* of the analogous normal stochastic process [5,7–9].

The description of superdiffusive processes within the same framework is still far from being completed. Whereas Lévy flights in the absence of an external force display a Lévy stable pdf, and therefore a diverging mean squared displacement (and thus could apply only to rather exotic physical processes) [1–3,5,10], *Lévy walks* (LWs) give a proper dynamical description in the superdiffusive domain. The temporal and spatial variables of LWs are strongly correlated, their steps being governed by a joint distribution $\psi(x,t)$, in which waiting time and step length pdfs, $\psi(t)$ and $\lambda(x)$ are no longer independent. In LWs, the occasional long jumps which are typical for Lévy flights are penalized through the introduction of a time cost. This spatiotemporal coupling can be achieved, in the simplest case, by the choice of $\psi(x,t) = \frac{1}{2}\psi(t)\delta(|x| - vt)$, i.e., by a constant velocity [3]. Such a model arises naturally when describing some limiting cases of molecular collisions [11]; its close relatives are reasonable candidates for describing turbulent dispersion [12]. A question of fundamental interest is therefore the formulation of LWs in terms of deterministic equations. Whereas previous approaches [13,14] in terms of fractional Klein-Kramers equations could reproduce the lower order moments of an LW, they were hampered by the fact that they could not describe the full pdf. The main complication on this way is the fact, that the overall LW process cannot be immediately considered as subordinated to a Wiener one (or to a simple random walk); however, as we proceed to show, it is exactly the strong correlation of the temporal and the spatial aspects of LWs which makes it possible to provide a description based on a process subordinated to a simple two-state Markovian process, cf. Ref. [15]. In the present work, we derive the *exact* deterministic evolution equation for LWs which holds both for the free motion and in a constant force field. The fact that the corresponding equation does not have a form of a Fokker-Planck or a Klein-Kramers equation explains the failure of previous attempts on the way of dynamical description of Lévy walks. In the following, we use rescaled quantities and concentrate on the one-dimensional case.

We first define a two-state Markovian random process describing the velocity switching and then proceed to generalize it to two different domains of LWs. Thus, let us denote by P_+ and P_- the probabilities to move to the right or to the left, respectively. Probability conservation demands that $P_+ + P_- = 1$. Moreover, for simplicity we assume that the absolute value of the velocity of motion to the right and to the

*Electronic address: igor.sokolov@physik.hu-berlin.de
†Electronic adddress: metz@nordita.dk

left is 1. Within the rate description, for a symmetric case, the probabilities $P_\pm$ satisfy the differential equation

$$\frac{d}{dt}P_\mp = P_\pm - P_\mp . \tag{1}$$

Equation (1) can be readily solved: Taking $P_- = 1 - P_+$ we get $(d/dt)P_+ = 1 - 2P_+$. The equilibrium situation corresponds to $P_+ = P_- = 1/2$, and the relaxation to this equilibrium from the initial condition $P_+ = 1$ is exponential,

$$P_\pm = \tfrac{1}{2} \pm \tfrac{1}{2}\exp(-2t). \tag{2}$$

Let us concentrate first on the switching process described by this equation: It is an alternating random process with the waiting-time pdf $\psi(t) = \exp(-t)$. At each ''tick'' the state is changed from $+1$ to -1 and back. $P_+(t)$ is then the probability that at time t, the state of the system is ''$+1$,'' i.e. that the overall number of full steps (changes of sign) was even. The Laplace transform of this probability is

$$P_+(u) = \sum_{n=0}^{\infty} \chi_{2n}(u) = \Psi(u) \sum_{n=0}^{\infty} \psi^{2n}(u) = \frac{1}{u[1+\psi(u)]}. \tag{3}$$

A similar expression for P_- involves the summation over the odd numbers of steps. In Eq. (3), the $\chi_{2n}(t)$ denote the probability that the walker performs $2n$ direction changes within an overall waiting time t. The probability of making no steps is $\chi_0(t) = \Psi(t) = 1 - \int_0^t \psi(t)dt$ with Laplace transform $\Psi(u) = u^{-1} - \psi(u)/u$ [3]. For our exponential function $[\psi(u) = 1/(1+u)]$, the result becomes $P_+(u) = 1/(2u) + 1/[2(2+u)]$, which is exactly the Laplace transform of Eq. (2).

Consider now a long-tailed waiting-time pdf $\psi(t) \sim t^{-1-\alpha}$ of the explicit form [16]

$$\psi(u) = \frac{1}{1+u^\alpha}, \quad 0<\alpha<1 \tag{4}$$

in Laplace space. This specific form has the following origin due to subordination [17]. In a system whose relaxation in its operational time is given by an exponential $\phi(\tau) = \exp(-\tau)$, we can introduce a coarse graining in which the operational time is divided into intervals $\Delta\tau$ (and $\Delta\tau$ taken as a new time unit). Assume that the duration of the physical time interval corresponding to $\Delta\tau$ is given by a one-sided Lévy distribution. The duration of the physical time corresponding to the interval τ is then a convolution of $n = \tau/\Delta\tau$ such distributions, and its Laplace transform is $\exp(-nu^\alpha)$. Averaging over n we get $\psi(u) \sim \int \exp(-nu^\alpha)\exp(-n)dn$, exactly reproducing Eq. (4). As an example we can explicitly determine $\psi(t)$ for $\alpha = \frac{1}{2}$: $\psi_{1/2}(t) = (\pi t)^{-1/2} - e^t \mathrm{erfc}\sqrt{t}$, with the asymptotic behaviors $t^{-1/2}$ $(t \ll 1)$ and $t^{-3/2}$ $(t \gg 1)$.

With waiting-time pdf (4), Eq. (1) generalizes to the fractional form

$$\frac{d}{dt}P_\mp = {}_0D_t^{1-\alpha}(P_\pm - P_\mp), \tag{5}$$

where ${}_0D_t^{1-\alpha} = (d/dt)\,{}_0D_t^{-\alpha}$, and ${}_0D_t^{-\alpha}$ is the fractional Riemann-Liouville integral operator defined in terms of

$${}_0D_t^{-\alpha}f(t) \equiv \frac{1}{\Gamma(\alpha)}\int_0^t dt' f(t')(t-t')^{\alpha-1}, \tag{6}$$

with the convenient property $\int_0^\infty e^{-ut}\,{}_0D_t^{-\alpha}f(t) = u^{-\alpha}f(u)$ [5,18]. In Eq. (5), the fractional derivative on the rhs describes a process which is subordinated to the simple exponential switching, being parametrized by the operational time τ; the subordination is defined by the Lévy stable waiting-time pdf. To see this let us compare two solutions: one using the ''CTRW'' time and another solving Eq. (5) directly.

From Eq. (5), with the initial conditions $P_+(0) = 1$ and $P_-(0) = 0$, we recover upon Laplace transformation

$$uP_+ - 1 = -u^{1-\alpha}P_+ + u^{1-\alpha}P_- ,$$

$$uP_- = u^{1-\alpha}P_+ - u^{1-\alpha}P_- ;$$

from the second equation, $P_- = 1/(u^\alpha + 1)P_+$, and therefore we find by insertion into the first,

$$P_+ = \frac{1+u^\alpha}{2u + u^{\alpha+1}}, \quad \text{and } P_- = \frac{1}{2u + u^{\alpha+1}}. \tag{7}$$

It is easy to verify that the same result is obtained by combining Eq. (4) with Eq. (3). Equation (7) describes the kinetics of moving to the left and to the right.

We now combine the purely temporal results for $P_\pm$ with the drift invoked by a constant velocity, distinguishing between two different cases. The ensuing propagator P of the associated symmetric random walk is combined from a superposition of two realizations of the switching process, taking place with the rate of 1/2 each, one in which the first step goes to the right, and one in which it goes left. This introduces an additional factor of 1/2 in all the following equations.

(i) Ballistic regime. In the Markovian case, the combination of process (2) with a velocity of magnitude 1 introduces the material derivatives $d_\pm \equiv (\partial/\partial t) \pm (\partial/\partial x)$. Viewing now $P_\pm$ as functions of x and t, the evolution equation for $P_\pm(x,t)$ result,

$$d_\pm P_\pm = \tfrac{1}{2}(P_\mp - P_\pm). \tag{8}$$

Both together produce the telegrapher's equation

$$\frac{\partial}{\partial t}P + \frac{\partial^2}{\partial t^2}P = \frac{\partial^2}{\partial x^2}P \tag{9}$$

also known as the Cattaneo equation [19], for the quantity of interest, the propagator $P = P_+ + P_-$. The Cattaneo equation describes a process which at short times behaves ballistically, $\langle x^2(t)\rangle \sim t^2$, and at long times exhibits normal diffusion, $\langle x^2(t)\rangle \sim t$. This derivation was based on the material derivatives $d_\pm$, whose Fourier-Laplace transform is $u \pm ik$. We now demonstrate that we reproduce exactly the propagator of an LW if we assume (*ad hoc*) that the corresponding fractional material derivative is defined through

$$\mathcal{F}_x\mathcal{L}_t\{d_\pm^{1-\alpha}f(x,t)\}=(u\pm ik)^{1-\alpha}f(k,u). \tag{10}$$

This choice is motivated by the fact that a waiting time is still coupled to a walk of length $x=t$, and that for anomalous transport processes the Fourier-Laplace space is the natural basis to introduce generalizations. Thus, we obtain

$$d_\pm P_\pm=\tfrac{1}{2}\,d_\pm^{1-\alpha}(P_\mp-P_\pm), \tag{11}$$

where the fractional material derivatives are to be interpreted in terms of Eq. (10). We solve Eq. (11) under the initial condition $P_+(x,0)=P_-(x,0)=\delta(x)/2$ so that, when introducing the propagator $P=P_++P_-$ and its counterpart $Q=P_+-P_-$, we have $P(x,0)=\delta(x)$ and $Q(x,0)=0$. With the abbreviations $\lambda_+=u+ik$ and $\lambda_-=\lambda_+^*=u-ik$, Eqs. (11) can be rewritten in terms of Fourier-Laplace transformed P and Q, as

$$\lambda_\pm(P\pm Q)-\tfrac{1}{2}=\mp\tfrac{1}{2}\,\lambda_\pm^{1-\alpha}Q. \tag{12}$$

The solution for P reads

$$P=\frac{\lambda_+^\alpha\lambda_-^{\alpha-1}+\lambda_-^\alpha\lambda_+^{\alpha-1}+\lambda_+^{\alpha-1}+\lambda_-^{\alpha-1}}{\lambda_+^\alpha+\lambda_-^\alpha+2\lambda_+^\alpha\lambda_-^\alpha}. \tag{13}$$

Let us show that this is an exact expression for the LW with waiting time pdf (4). To this end, note that within the CTRW the propagator is obtained as $P(k,u)=\Psi(k,u)/[1-\psi(k,u)]$ in the case of spatiotemporal coupling with $\psi(x,t)=\frac{1}{2}[\delta(x-t)+\delta(x+t)]\psi(t)$ and $\Psi(x,t)=\frac{1}{2}[\delta(x-t)+\delta(x+t)]\Psi(t)$ [20]. Consequently, we find

$$\psi(u,k)=\tfrac{1}{2}\,[\psi(u+ik)+\psi(u-ik)]$$

and an analogous expression for $\Psi(u,k)$, such that we arrive at the Fourier-Laplace form of P:

$$P=\frac{[1-\psi(\lambda_+)]/\lambda_++[1-\psi(\lambda_-)]/\lambda_-}{2-\psi(\lambda_+)-\psi(\lambda_-)}. \tag{14}$$

With $\psi(u)$ given by Eq. (4), Eq. (13) is reproduced and we have shown that Eq. (12) with definition (10) describes an LW. From representation (14), we find the Laplace space form $\langle x^2(u)\rangle=2(u^\alpha+1-\alpha)/(u^3+u^{3+\alpha})$ of the second moment, from which we obtain the limiting behaviors $\langle x^2(t)\rangle\sim t^2$ for $t\ll 1$ and $\langle x^2(t)\rangle\sim(1-\alpha)t^2$ for $t\gg 1$, i.e., a mere decrease in the amplitude of an *overall ballistic* process: the memory which we introduced by the long-tailed form of ψ leads to an extreme persistence in a given direction on all time scales. There is no turnover to a process with a smaller exponent of t as in the Cattaneo case.

(ii) Subballistic regime. Let us now compare this with the better known case of the subballistic domain. We again follow our above obtained recipe of formulating two equations for the direction-switching process with the waiting-time distribution of interest, and then change the time derivatives for the material ones. There is an heuristic way immediately leading to the equations: in the Laplace representation, the system of equations

$$uP_+-1=\tfrac{1}{2}\,f(u)(-P_++P_-),$$
$$uP_-=\tfrac{1}{2}\,f(u)(P_+-P_-) \tag{15}$$

leads to the solution $P_+=[2(u+f(u))]/[u(u+2f(u))]$. Noting that according to Eq. (3), this should correspond to $P_+=\{u[1+\psi(u)]\}^{-1}$, we find the relation $\psi(u)=1/[1+u/f(u)]$. If we want a function behaving for small u as $\psi(u)\sim 1-u-u^{1+\beta}$ (i.e., one with $\alpha=1+\beta$ being in the interval between 1 and 2), we have to choose $f(u)=1+u^\beta$, such that $\psi(u)=(1+u^\beta)/(1+u^\beta+u)$ is the desired pdf. Following along the lines of case (i), we infer the equations

$$\frac{d}{dt}P_\pm=\tfrac{1}{2}\,(1+{}_0D_t^\beta)(P_\mp-P_\pm) \tag{16}$$

for the alternating process, and we obtain the equation with material derivatives,

$$d_\pm P_\pm=\tfrac{1}{2}\,(1+d_\pm^\beta)(P_\mp-P_\pm) \tag{17}$$

for the Lévy walk. Using our formal rules, we find in the Fourier-Laplace representation:

$$\tfrac{1}{2}\,\lambda_\pm(P\pm Q)-\tfrac{1}{2}=\mp\tfrac{1}{2}\,(1+\lambda_+^\beta)Q,$$

whose solution is

$$P=\frac{2+\lambda_+^\beta+\lambda_-^\beta+\lambda_++\lambda_-}{2\lambda_+\lambda_-+\lambda_+(1+\lambda_-^\beta)+\lambda_-(1+\lambda_+^\beta)}. \tag{18}$$

This corresponds exactly to Eq. (14) for the new ψ and again corroborates the recipe to generalize $d_\pm$ to the fractional material derivatives $d_\pm^\beta$ in the Fourier-Laplace domain. The second moment of this process is obtained as $\langle x^2(u)\rangle=2(u+\beta u^\beta)/(u^4+u^{3+\beta}+u^3)$, giving rise to the limiting behaviors $\langle x^2(t)\rangle\sim t^2$ for $t\ll 1$ and $\langle x^2(t)\rangle\sim 2\beta t^{2-\beta}/\Gamma(3-\beta)$ for $t\gg 1$, i.e., a transition from initial ballistic to terminal subballistic superdiffusive behavior, in an analogy to the CTRW result [3,20].

Let us now discuss the coordinate-time representation of the fractional material derivative. Using the well-known relation $\mathcal{L}^{-1}\{F(u+b)\}=e^{-bt}f(t)$ of the Laplace transformation, we obtain after some steps

$$d_\pm^\alpha P(x,t)={}_0D_t^\alpha P(x\pm t,t). \tag{19}$$

That is, the fractional material derivative generalizes the regular material derivative, $d_\pm P(x,t)=(d/dt)P(x\pm t,t)\equiv(\partial/\partial t\pm\partial/\partial x)P$ for $\alpha=1$, through the introduction of the standard (acting on t only) Riemann-Liouville operator acting on the entire right hand side.

For subdiffusion, the major advantage of the fractional dynamical equation formulation is in the possibility to easily generalize to situations with an external force field, which led to the fractional Fokker-Planck equation [5,6]. Here, we start with incorporating a constant external force. To this end, let us consider the physical realization of the walk in a splitting flow: in the upper half plane, the particle moves to the right; $v_x^+ = v_0$ for $y>0$, in the lower half plane it moves to the left: $v_x^- = -v_0$ for $y<0$. The motion in the y direction dictates the waiting-time distribution. If it is a simple diffusion, the overall process is a Lévy walk with $\alpha = 1/2$ [1]. Imagine now, we have a force acting in the x direction. The force causes a sliding of the particle with respect to the flow, so that now $v_x^+ = v_0 + \mu f$ for $y>0$ and $v_x^- = -v_0 + \mu f$ for $y<0$. This corresponds to changing $d_\pm^\alpha$ to the constructs corresponding to

$$d_{f,\pm}^\alpha P(x,t) = {}_0D_t^\alpha P(x + (\mu f \pm v)t, t), \tag{20}$$

whose Fourier-Laplace transform produces

$$d_{f,\pm}^\alpha \rightarrow [u + i(\mu f \pm v)k]^\alpha. \tag{21}$$

This can be verified by the comparison of the CTRW results for a constant force [6]. That is, in both the force-free and the constant force cases, we observe some type of generalized d'Alembert principle reflecting the δ coupling of x and t [21]. Translating the dynamic equations (10) and (17) with the fractional material derivatives into an equation for the propagator P produces a rather complicated expression. This can be circumvented by a different definition of the fractional operators, as shown in Ref. [22]. However, the latter does not allow for the incorporation of a bias and is thus not suited for our purpose.

In our approach we were guided by the equivalence between position x and time t in the LW framework, enforced by the δ-coupling $\psi(x,t) = \frac{1}{2}\delta(|x| - t)\psi(t)$, which could in fact be rewritten in terms of the jump length distribution with the appropriate long-tailed form for λ [3]. This equivalence gives rise to the occurrence of the material derivative, in complete analogy to the Brownian Cattaneo case. However, in the presence of long-tailed temporal correlations of the kind $\psi(t) \sim t^{-1-\alpha}$, the fractional variant of the material derivative emerges, with its simple representations in both Fourier-Laplace and (x,t) domains. This treatment is amenable for the case of a constant external force. Whether there is a similar treatment for arbitrary force $f(x)$ is not clear at present. The representation of LWs in terms of the left-right processes $P_\pm$ reveals a surprisingly simple structure for the generalization of the material derivative, and therefore we are confident that it is the right direction towards including a general external force $f(x)$.

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$$d_{f(x),\pm}^\alpha P(x,t) = {}_0D_t^\alpha P(x(t),t) = {}_0D_t^\alpha P\left(x \pm vt + \int_0^t \mu f(x(t))dt, t\right);$$
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